

G-SYSTEMS AND DEFORMATION OF G-ACTIONS ON \mathbb{R}^d

BENOIT DHERIN AND IGOR MENCATTINI

ABSTRACT. Given a (smooth) action φ of a Lie group G on \mathbb{R}^d we construct a DGA whose Maurer–Cartan elements are in one-to-one correspondence with some class of deformations of the (induced) G -action on $C^\infty(\mathbb{R}^d)[[\hbar]]$. In the final part of this note we discuss the cohomological obstructions to the existence and to the uniqueness (in a sense to be clarified) of such deformations.

1. INTRODUCTION

In this note we introduce an algebra of formal differential operators which will be obtained from a class of Fourier Integral Operators (FIO from now on) in the limit $\hbar \rightarrow 0$. Given a (smooth) G -action on \mathbb{R}^d we will use these formal operators to define a family of *deformations* (in a sense will be explained below) of the induced G -action on the space $C^\infty(\mathbb{R}^d)[[\hbar]]$. These families of deformations are controlled by the Maurer–Cartan elements of a Differential Graded Algebra (DGA since now on) of *formal amplitudes*. Such a DGA will be the main ingredient for the analysis of the cohomological obstruction to the existence and to the uniqueness (rigidity) of such deformations. These families of deformations and the corresponding DGA play an important role in the theory of the *quantum momentum maps* as it is explained in the paper [2].

2. A CLASS OF FIOS, THEIR ASYMPTOTIC EXPANSIONS AND THE FORMAL OPERATORS

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of Schwartz functions on \mathbb{R}^d , see for example [9]. In this paper we will be concerned with the following class of FIO:

$$(2.1) \quad \text{Op}(a, \varphi)\psi(x) = \int_{\mathbb{R}^d} \widehat{\psi}(\bar{\xi})a(x, \bar{\xi})e^{\frac{i}{\hbar}\langle \bar{\xi}, \varphi^{-1}(x) \rangle} \frac{d\bar{\xi}}{(2\pi\hbar)^{\frac{d}{2}}}$$

where $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\widehat{\psi}$ is its *semiclassical Fourier transform*, see [6] or [10], $a(x, \xi) \in S^0$ and $\varphi \in \mathbf{Diff}_b(\mathbb{R}^d)$, see Remark 1 below.

Remark 1. S^0 is the space of C^∞ -functions (defined on \mathbb{R}^d) whose elements depend on a parameter $\hbar \in (0, \hbar_0]$ and which satisfy the following differential inequality

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq A_{\alpha, \beta} (1 + |\xi|)^{-|\alpha|}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^d$, see for example [8]. In what follows the \hbar -dependence will be not, in general, explicitly written. Finally, $\varphi \in \mathbf{Diff}_b(\mathbb{R}^d)$ is the group of *bounded* diffeomorphisms of \mathbb{R}^d , i.e. the subgroup of $\mathbf{Diff}(\mathbb{R}^d)$ whose elements φ are such that $\sup_{x \in \mathbb{R}^d} |\partial_x^\beta \varphi(x)| < \infty$ for all multi-indices $\beta \in \mathbb{N}^d \setminus (0, \dots, 0)$.

Remark 2. Even if in this letter we will mainly interested in the formal properties of the operators defined in formula (2.1), we remind that they are a class of continuous operators on $\mathcal{S}(\mathbb{R}^d)$, which is closed with respect to composition: this means that given $\text{Op}(a, \varphi), \text{Op}(b, \varphi_2)$ as above, their composition $\text{Op}(a, \varphi) \circ \text{Op}(b, \varphi_2)$ can be represented as $\text{Op}(c, \varphi)$ for suitable $c \in S^0$, and $\mathbf{Diff}_b(\mathbb{R}^d)$.

In the following Example we introduce a simple, though interesting, family of such operators.

Example 3 (ξ -independent symbols). In what follows, with some abuse of notation we will write the FIO of formula (2.1) as

$$\text{Op}(a, \varphi)\psi(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(\bar{x}) a(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \varphi^{-1}(x) - \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^d}$$

If $a = a(x)$ then

$$\text{Op}(a, \varphi)\psi(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(\bar{x}) a(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \varphi^{-1}(x) - \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^d} = a(x) \psi(\varphi^{-1}(x))$$

Computing $\text{Op}(a, \varphi_1) \circ \text{Op}(b, \varphi_2)\psi(x)$ we arrive to the following formula

$$\text{Op}(a, \varphi_1) \circ \text{Op}(b, \varphi_2) = \text{Op}(a(x)b(\varphi^{-1}(x)), \varphi_1 \circ \varphi_2)$$

We now work out the asymptotic expansion of the bounded operators (2.1) in the limit $\hbar \rightarrow 0$. First, we fix the dependence in \hbar for the amplitude a as follows:

$$(2.2) \quad a(x, \xi) = a^0(x, \xi) + a^1(x, \xi)\hbar + a^2(x, \xi)\hbar^2 + \dots,$$

where the $a^n \in S^0$ do not depend on \hbar for all n . Namely, the Borel summation lemma (see [6, Prop. 2.3.2, p. 14] for instance) guarantees then that there exists an amplitude in S^0 depending on \hbar whose asymptotic expansion in \hbar yields back (2.2).

Now, changing the variable $\tilde{\xi} = \xi/\hbar$ and letting $\hbar \rightarrow 0$ (which allows us to compute a Taylor series of the amplitude at $(x, 0)$), we obtain that:

$$\begin{aligned} \text{Op}(a(x, \xi), \varphi)\psi(x) &= \int \psi(\bar{x}) a(x, \hbar\tilde{\xi}) e^{i \langle \tilde{\xi}, \varphi^{-1}(x) - \bar{x} \rangle} \frac{d\bar{x} d\tilde{\xi}}{(2\pi)^d}, \\ &= \sum_{n \geq 0} \hbar^n \text{Op}_1(P^n, \varphi)\psi(x). \end{aligned}$$

where Op_1 is the same integral operator as Op except with the parameter \hbar in the phase set to 1, and where

$$P^n(x, \xi) = \sum_{|\alpha| \leq n} f_\alpha(x) \xi^\alpha,$$

are polynomial in ξ of order n with coefficients in $S_d(1)$, the space of C^∞ -functions on \mathbb{R}^d , which are bounded with all their derivative bounded. (actually, $f_\alpha(x) = \frac{1}{|\alpha|!} \partial_\xi^\alpha a_{n-|\alpha|}(x, 0)$).

Since, for a polynomial $P^n(x, \xi)$ in ξ as above, the corresponding operator

$$\text{Op}_1(P^n, \varphi)\psi(x) = \sum_{|\alpha| \leq n} f_\alpha(x) (D_x^\alpha \psi)(\varphi^{-1}(x)) = (P^n(x, D) \psi)(\varphi^{-1}(x))$$

is a differential operator of order n (composed with the pullback), we obtain for $\text{Op}(a, \varphi)$ an asymptotic expansion in terms of infinite order differential operators of the form:

$$(2.3) \quad \text{Op}(a, \varphi)\psi(x) = P^0(x)\psi(\varphi^{-1}(x)) + \sum_{n \geq 1} \hbar^n (P^n(x, D) \psi)(\varphi^{-1}(x)).$$

Remark 4. This derivation for the asymptotic (2.3) is a shortcut for the usual stationary phase expansion. One recovers (2.3) by using the usual stationary phase expansion (see [10]) for quadratic phase using the following change of variable $\bar{y} = \varphi^{-1}(x) - \bar{x}$.

In the following definition, we retain only the formal aspects of the asymptotics, forgetting that the operators (2.1) are actually bounded operators (i.e. the amplitudes are in S^0 and the action is in $\mathbf{Diff}_b(\mathbb{R}^d)$).

Definition 5. We define the algebra \mathcal{D} of formal operators of the form

$$(2.4) \quad \text{Op}_1(P, \varphi)\psi(x) = P^0(x)\psi(\varphi^{-1}(x)) + \sum_{n \geq 1} \hbar^n (P^n(x, D) \psi)(\varphi^{-1}(x)), \quad \varphi \in \mathbf{Diff}(\mathbb{R}^d)$$

which acts on the formal space of functions $C^\infty(\mathbb{R}^d)[[\hbar]]$, and where

$$P^n(x, D) = \sum_{|\alpha| \leq n} f_\alpha(x) D^\alpha,$$

is a differential operator of order n with coefficients $f_\alpha \in C^\infty(\mathbb{R}^d)$. The corresponding space of symbols \mathcal{P} is the space of formal functions of the form

$$P(x, \xi) = P^0(x) + \sum_{n \geq 1} \hbar^n \sum_{|\alpha| \leq n} f_\alpha(x) \xi^\alpha,$$

with $P^0(x), f_\alpha \in C^\infty(\mathbb{R}^d)$.

Remark 6. By a direct computation of $\text{Op}_1(P, \varphi_1) \circ \text{Op}_1(K, \varphi_2)$ applied to a (formal) series $\psi \in C^\infty(\mathbb{R}^d)[[\hbar]]$, we can get a close formula for the (formal) symbol representing such an operator. We will not write the explicit expression for such a formal symbol, but we will denote it as $P_{\varphi_1} \star_{\varphi_2} K$ (to remind its dependence on the two diffeomorphisms φ_1, φ_2). In this way, the composition law in \mathcal{D} will be written as:

$$(2.5) \quad \text{Op}_1(P, \varphi_1) \circ \text{Op}_1(K, \varphi_2) = \text{Op}_1(P_{\varphi_1} \star_{\varphi_2} K, \varphi_1 \circ \varphi_2).$$

Remark 7. Note that formula (2.5) is the binary operation of a semigroup structure on $\mathcal{P} \times \mathbf{Diff}(\mathbb{R}^d)$.

3. DGAS OF (FORMAL) G -AMPLITUDES

Given G as above we fix $\varphi : G \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ to be a smooth (left) action. Using φ we define a Differential Graded Algebra (or DGA for short), whose (homogeneous) elements of degree k are, roughly speaking, formal amplitudes depending on k group variables, and whose (graded) product corresponds to the composition of the formal operators of the type (2.4). More precisely

Definition 8. The complex of formal G -amplitudes $(\mathcal{P}_\varphi^\bullet, d)$ is the space of k -cochains

$$\mathcal{P}_\varphi^k = \{a : G \times \cdots \times G \rightarrow \mathcal{P}\}, \quad \mathcal{P}_\varphi^0 = \mathcal{P}$$

endowed with the differential $d : \mathcal{P}_\varphi^k \rightarrow \mathcal{P}_\varphi^{k+1}$ defined by:

$$(3.1) \quad (da)(g_1, \dots, g_k) = \sum_{i=1}^k (-1)^i a(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}),$$

which we extend by \mathbb{C} -linearity to $\mathcal{P}_\varphi^\bullet = \bigoplus_{k \geq 0} \mathcal{P}_\varphi^k$. The complex is normalized by the condition $a_{e, \dots, e} = 1$ with $e \in G$ being the group unit.

For all k, l we define

$$\star : \mathcal{P}_\varphi^k \times \mathcal{P}_\varphi^l \rightarrow \mathcal{P}_\varphi^{k+l}$$

and we then extend it by \mathbb{C} -linearity to $\mathcal{P}_\varphi^\bullet$. We remind now that:

Definition 9. If (A, \star, d) is a DGA then:

- (1) The solutions of the **Maurer-Cartan equation** $da + a \star a = 0$ are called **Maurer-Cartan elements**. The set of all Maurer-Cartan elements of A , which is a subset of A^1 , will be denoted by $\text{MC}(A)$.
- (2) Two Maurer-Cartan elements $a, b \in \text{MC}(A)$ are called **gauge equivalent** if there exists an invertible $u \in A^0$ such that $au - ua = du$.

Theorem 10. Given $(\mathcal{P}_\varphi^\bullet, d, \star)$ as above, then:

- (1) $(\mathcal{P}_\varphi^\bullet, d, \star)$ is a DGA.
- (2) The Maurer-Cartan elements in $\mathcal{P}_\varphi^\bullet$ are in one-to-one correspondence with the representations of G of the form $g \rightsquigarrow \text{Op}(a, \varphi_g)$. Moreover, gauge equivalent Maurer-Cartan elements yields equivalent representations.

Proof. Let $a \in \mathcal{P}_\varphi^k$, $b \in \mathcal{P}_\varphi^l$. and let $a_{g_1, \dots, g_k} \star a_{g_{k+1}, \dots, g_{k+l}}$ be a short hand notation to denote the product of amplitudes introduced in formula (2.5), i.e.

$$a_{g_1, \dots, g_k} \star b_{g_{k+1}, \dots, g_{k+l}} = \left(a(g_1, \dots, g_k) \right)_{\varphi_{g_1 \dots g_k}} \star_{\varphi_{g_{k+1} \dots g_{k+l}}} \left(b(g_{k+1}, \dots, g_{k+l}) \right)$$

Then part (1) will follow once we will prove that (a) d squares to zero, (b) \star is associative and finally that (c) d is a derivation of \star . The part (a) is a standard computation, while part (b) follows from the associativity of the composition of formal FIOs corresponding to the symbols a_{g_1, \dots, g_k} for all $k \geq 0$, i.e. i.e.

$$(3.2) \quad \text{Op}(a_{g_1, \dots, g_k}, \varphi_{g_1 \dots g_k}) \circ \text{Op}(b_{g_{k+1}, \dots, g_{k+l}}, \varphi_{g_{k+1} \dots g_{k+l}}) = \text{Op}(a_{g_1, \dots, g_k} \star b_{g_{k+1}, \dots, g_{k+l}}, \varphi_{g_1 \dots g_{k+l}})$$

For part (c) we compute:

$$\begin{aligned}
\delta(a \star b)_{g_1, \dots, g_{k+l+1}} &= \sum_{i=1}^{k+l} (-1)^i (a \star b)_{g_1, \dots, g_i g_{i+1}, \dots, g_{k+l+1}}, \\
&= \left(\sum_{i=1}^k (-1)^i a_{g_1, \dots, g_i g_{i+1}, \dots, g_k} \right) \varphi_{g_1 \dots g_k} \star_{\varphi_{g_{k+1} \dots g_{k+l}}} b_{g_{k+1}, \dots, g_{k+l+1}} \\
&\quad + (-1)^k a_{g_1, \dots, g_k} \varphi_{g_1 \dots g_k} \star_{\varphi_{g_{k+1} \dots g_{k+l}}} \left(\sum_{i=1}^l (-1)^i b_{g_k, \dots, g_{k+i-1} g_{k+i}, \dots, g_{k+l+1}} \right) \\
&= ((\delta a) \star b)_{g_1, \dots, g_{k+l+1}} + (-1)^k (a \star (\delta b))_{g_1, \dots, g_{k+l+1}}
\end{aligned}$$

Let us now prove (2) of the Theorem. Let $a \in \mathcal{P}_\varphi^1$. Then $g \rightsquigarrow \text{Op}(a_g, \varphi_g)$ is a representation if and only if $\text{Op}(a_{g_1 g_2}, \varphi_{g_1 g_2}) = \text{Op}(a_{g_1}, \varphi_{g_1}) \circ \text{Op}(a_{g_2}, \varphi_{g_2})$ for all $g_1, g_2 \in G$. But the right hand side of this equality is $\text{Op}(a_{g_1} \star a_{g_2}, \varphi_{g_1} \varphi_{g_2})$. This means that $g \rightsquigarrow \text{Op}(a_g, \varphi_g)$ is a representation if and only if:

$$\text{Op}(a_{g_1 g_2} - da_{g_1, g_2}, \varphi_{g_1 g_2}) = 0$$

which is equivalent to say that $a \in \mathcal{P}_\varphi^1$ is a Maurer–Cartan element of $(\mathcal{P}_\varphi^\bullet, d, \star)$. Finally let us check now that two gauge equivalent Maurer–Cartan elements induce equivalent representations. First, we note that in $\mathcal{P}_\varphi^\bullet$, all elements of degree zero are cocycles. This means that $a, b \in \text{MC}(\mathcal{P}_\varphi^\bullet)$ are gauge equivalent if there is an invertible $u \in \mathcal{P}_\varphi^0$ such that $au = ua$. Since u is of degree zero, neither u nor $\text{Op}(u)$ depend on group variables. The formula $au = ua$ implies that the corresponding operators satisfy

$$\text{Op}(a_g, \varphi_g) \circ \text{Op}(u) = \text{Op}(u) \circ \text{Op}(a_g, \varphi_g), \quad g \in G$$

That is, $\text{Op}(u)$ intertwines the two representations; since $\text{Op}(u)$ is invertible, because u is invertible, the representations $g \rightsquigarrow \text{Op}(a_g, \varphi_g)$ and $g \rightsquigarrow \text{Op}(b_g, \varphi_g)$ are equivalent. \square

Because of the importance of the Maurer–Cartan elements of the complex of G -amplitudes, we will give to them the following name:

Definition 11 (G -system). A G -system of the complex of formal amplitudes $(\mathcal{P}_\varphi^\bullet, d, \star)$ is an element of $\text{MC}(\mathcal{P}_\varphi^\bullet)$.

Proposition 12. Let $a = P^0(x) + \hbar P^1 + \dots \in \mathcal{P}_\varphi^1$ be a G -system in $\mathcal{P}_\varphi^\bullet$.

- (1) Then P^0 is a Maurer–Cartan element in $\mathcal{P}_\varphi^\bullet$.
- (2) It defines a new differential on $\mathcal{P}_\varphi^\bullet$ as follows:

$$(3.3) \quad d_{P^0} a = da + [P^0, a] = da + P^0 \star a - (-1)^{|a|} a \star P^0.$$

- (3) Moreover, P^1 is a cocycle with respect to this new differential, and we get the following recursive equations for the higher order terms

$$(3.4) \quad d_{P^0} P^n = - \sum_{\substack{i+j=n \\ i, j \geq 1}} P^i \star P^j.$$

Proof. The Maurer–Cartan equation at order zero in \hbar reads:

$$dP^0 + P^0 \star P^0 = 0,$$

which means that P^0 is itself a Maurer–Cartan element. Now it is a general fact that a differential d twisted by a Maurer–Cartan element as in (3.3) is again a differential.

The Maurer–Cartan equation at order 1 in \hbar reads

$$dP^1 + P^0 \star P^1 + P^1 \star P^0 = 0,$$

which is exactly $d_{P^0}P^1 = 0$ because P^1 is of degree 1 (it has only one group variable). At last, we obtain (3.4) by looking at the Maurer–Cartan equation at order $n \geq 2$. \square

We conclude now this Section going back to the Example 3:

Example 13. A ξ -independent G -system in $(\mathcal{P}_\varphi^\bullet, d, \star)$ satisfies the condition $a_{g_1 g_2}(x) = a_{g_1}(x) a_{g_2}(\varphi_{g_1}^{-1}(x))$ for all $g_1, g_2 \in G$ and $x \in \mathbb{R}^d$. We observe that this is the *multiplicative* analogue of being a 1-cocycle of G with values on $C^\infty(\mathbb{R}^d)$ (G acts on $C^\infty(\mathbb{R}^d)$ by pullback). It is probably worthwhile to observe that it is possible to *linearize* the condition in (2) as follows. Let us consider ξ -independent element $a \in \mathcal{P}_\varphi^1$ of the form $g \rightsquigarrow a_g(x) = e^{iS_g(x)}$ where $S(x)$ is smooth function. It is a simple computation to show that a is a G -system if and only if

$$S_{g_1 g_2}(x) = S_{g_1} + S_{g_2}(\varphi_{g_1}^{-1}(x))$$

for all $g_1, g_2 \in G$ and $x \in \mathbb{R}^d$. In other words, $g \rightsquigarrow e^{iS_g(x)}$ is a G -system if and only if $S(x)$ is a 1-cocycle of G with values in $C^\infty(\mathbb{R}^d)$ (still the G -action on $C^\infty(\mathbb{R}^d)$ is by pullback). Let us show now that if $S - \tilde{S} = \delta K$, where S and \tilde{S} are 1-cocycle and K is a 0-cochain of G with values in $C^\infty(\mathbb{R}^d)$, then the induced representations (of G) $g \rightsquigarrow \text{Op}(e^{iS_g(x)}, \varphi_g)$ and $g \rightsquigarrow \text{Op}(e^{i\tilde{S}_g(x)}, \varphi_g)$ are equivalent. Consider the multiplication operator $\hat{K}\psi(x) = e^{iK(x)}\psi(x)$. Then:

$$\begin{aligned} (\text{Op}(e^{iS_g(x)}, \varphi_g) \circ \hat{K})\psi(x) &= e^{i(S_g(x) + K(\varphi_g^{-1}(x)))}\psi(\varphi_g^{-1}(x)), \\ (\hat{K} \circ \text{Op}(e^{i\tilde{S}_g(x)}, \varphi_g))\psi(x) &= e^{i(\tilde{S}_g(x) + K(x))}\psi(\varphi_g^{-1}(x)). \end{aligned}$$

Therefore, the relation $\tilde{S}_g(x) - S_g(x) = K(\varphi_g^{-1}(x)) - K(x) = (\delta K)_g(x)$ implies that $\text{Op}(e^{iS_g(x)}, \varphi_g) \circ \hat{K} = \hat{K} \circ \text{Op}(e^{i\tilde{S}_g(x)}, \varphi_g)$. From this we can conclude that $H^1(G, C^\infty(\mathbb{R}^d))$ controls the deformations of the G -action φ of the form $g \rightsquigarrow \text{Op}(e^{iS_g(x)}, \varphi_g)$. In particular, if this cohomology group vanishes, all such deformations are equivalent to the trivial one.

4. EXISTENCE AND RIGIDITY THEOREM

In this Section, we give cohomological conditions for the existence of formal G -systems, that is, Maurer–Cartan elements in \mathcal{P}_φ . The discussion that follows is based on

appendix A of [1]. The main fact is that $\mathcal{P}_\varphi^\bullet$ is a *complete* DGA in the sense of [1]; complete DGA have neat cohomological conditions governing the existence and obstruction of Maurer–Cartan elements.

Definition 14. We define $\mathbf{Pol}_d(n)$ for $n \geq 0$ to be the space of polynomials in ξ of the form

$$P(x, \xi) = \sum_{|\alpha| \leq n} f_\alpha(x) \xi^\alpha,$$

where $f_\alpha \in C^\infty(\mathbb{R}^d)$.

First of all, $\mathcal{P}_\varphi^\bullet$ has a natural filtration

$$\dots \subset F^{k+1}\mathcal{P}_\varphi^\bullet \subset F^k\mathcal{P}_\varphi^\bullet \subset \dots \subset F^1\mathcal{P}_\varphi^\bullet \subset F^0\mathcal{P}_\varphi^\bullet = \mathcal{P}_\varphi^\bullet,$$

for which each of the $(F^k\mathcal{P}_\varphi^\bullet, d)$ is a subcomplex and such that

$$\star : F^k\mathcal{P}_\varphi^\bullet \times F^l\mathcal{P}_\varphi^\bullet \rightarrow F^{k+l}\mathcal{P}_\varphi^\bullet$$

This filtration is given by

$$F^k\mathcal{P}_\varphi^\bullet = \left\{ \sum_{n \geq k} h^n P^n : P^n \in \mathbf{Pol}_d(n) \right\}, \quad k \geq 1.$$

We have then a tower

$$\mathcal{P}_\varphi^\bullet / F^1\mathcal{P}_\varphi^\bullet \leftarrow \mathcal{P}_\varphi^\bullet / F^1\mathcal{P}_\varphi^\bullet \leftarrow \dots,$$

whose inverse limit is exactly $\mathcal{P}_\varphi^\bullet$. This makes $\mathcal{P}_\varphi^\bullet$ a **complete** DGA in the sense of the Appendix A of [1].

Definition 15. Define the graded vector space $\mathbf{Pol}_d^\bullet(n)$ to be

$$\mathbf{Pol}_d^\bullet(n) := \{P : G^k \rightarrow \mathbf{Pol}_d(n)\}, \quad n, k \geq 0.$$

Observe that, as graded vector space, we have that

$$(4.1) \quad \mathcal{F}^n\mathcal{P}_\varphi^\bullet / \mathcal{F}^{n+1}\mathcal{P}_\varphi^\bullet \simeq \mathbf{Pol}_d^\bullet(n)$$

and the following decomposition of the complex of formal G -amplitudes:

$$\mathcal{P}_\varphi^\bullet = \mathbf{Pol}_d^\bullet(0) \oplus \hbar \mathbf{Pol}_d^\bullet(1) \oplus \hbar^2 \mathbf{Pol}_d^\bullet(2) \oplus \dots$$

Let $P^0 \in \mathbf{Pol}_d^\bullet(0)$ be a Maurer–Cartan element. Then the twisted differential d_{P^0} defined by formula (3.3), respects this decomposition and $(\mathbf{Pol}_d^\bullet(n), d_{P^0})$ is a complex for each $n \geq 0$. These complexes will be the main ingredients in our existence and rigidity results for formal G -systems.

From Proposition 12, we get that if

$$(4.2) \quad P^0 + \hbar P^1 + \hbar^2 P^2 + \dots$$

is a Maurer–Cartan element, then P^0 is a Maurer–Cartan element in $\mathcal{P}_\varphi^\bullet$ and P^1 is a 1-cocycle in $(\mathbf{Pol}_d^\bullet(1), d_{P^0})$. Now if we start with a Maurer–Cartan element P^0 and a 1-cocycle P^1 , in general $P^0 + \hbar P^1$ is not a Maurer–Cartan element in $\mathcal{P}_\varphi^\bullet$, and we may wonder whether it is possible to find higher terms to get a Maurer–Cartan element.

Another question is whether the representation obtained from (4.2) is equivalent to the one obtained by the first term only, i.e. when a Maurer-Cartan element is gauge equivalent to its first term.

Definition 16. A Maurer-Cartan element P^0 in $\mathcal{P}_\varphi^\bullet$ is called rigid if all Maurer-Cartan elements having as first term P^0 are gauge equivalent to this first term.

In term of the induced representations, P^0 being rigid means that all the representations obtained from Maurer-Cartan elements of the form (4.2) are equivalent as representations to the representation

$$\text{Op}(P^0, \varphi_g)\psi(x) = P^0(x)\psi(\varphi_g^{-1}(x)).$$

The following theorem gives cohomological conditions answering the questions mentioned above.

Theorem 17. Let $P^0 \in \mathbf{Pol}_d^\bullet(0)$ be a Maurer-Cartan element and $P^1 \in \mathbf{Pol}_d^1(1)$ a 1-cocycle (i.e. $d_{P^0}P^1 = 0$). If

$$H^2(\mathbf{Pol}_d^\bullet(n), d_{P^0}) = 0, \quad n \geq 2,$$

then there exists a Maurer-Cartan element ω in $\mathcal{P}_\varphi^\bullet$ such that

$$\omega = P^0 + \hbar P^1 + \mathcal{O}(\hbar^2).$$

Moreover if

$$H^1(\mathbf{Pol}_d^\bullet(n), d_{P^0}) = 0, \quad n \geq 1,$$

the Maurer-Cartan element P^0 is rigid.

Proof. The proof relies on Proposition A.3 and A.6 of the Appendix A of [1]. Since $\gamma = P^0 + \hbar P^1$ is a Maurer-Cartan element modulo $\mathcal{F}^2\mathcal{P}_\varphi^\bullet$, Proposition A.3 tells us that there exist a Maurer-Cartan element $\omega = P^0 + \hbar P^1 + \mathcal{O}(\hbar^2)$ provided

$$H^2(\mathcal{F}^n\mathcal{P}_\varphi^\bullet/\mathcal{F}^{n+1}\mathcal{P}_\varphi^\bullet, d_\gamma) = 0, \quad n \geq 2,$$

where d_γ is the operator $d_\gamma a = da + [\gamma, a]$, which becomes a differential on the quotient $\mathcal{F}^n\mathcal{P}_\varphi^\bullet/\mathcal{F}^{n+1}\mathcal{P}_\varphi^\bullet$. The first part of the theorem follows from (4.1) and the fact that d_γ becomes d_{P^0} when passing to the quotient (because P^1 has one power of \hbar , which will make this term disappear in the quotient). The rigidity part of the theorem is a direct application of Proposition A.6 with the same observations as above. \square

4.1. Trivial action. We will consider now the case $\varphi_g = \text{id}$ for all $g \in G$ (i.e. the group G acts trivially on \mathbb{R}^d) and we further suppose that the first term of the deformation is also trivial, i.e. we consider G -systems of the form

$$a_g = 1 + \hbar P_g^1 + \hbar^2 P_g^2 + \dots$$

Under these assumptions the representations of G given by $g \rightsquigarrow \text{Op}(a_g, \varphi_g)$ are deformations of the trivial representation of G on $C^\infty(\mathbb{R}^d)[[\hbar]]$ and they are of the form

$$(4.3) \quad \text{Op}(a_g, \varphi_g)\psi(x) = \text{id} + \sum_{n \geq 1} \hbar^n P_g^n(x, D),$$

where $P_g^n(x, D)$ is a differential operator of order n with non-constant bounded coefficients.

Let us endow $C^\infty(\mathbb{R}^d)$ with a trivial G -bimodule structure and let $H^\bullet(G, C^\infty(\mathbb{R}^d))$ be the corresponding cohomology. Then:

Theorem 18. *If $H^2(G, C^\infty(\mathbb{R}^d)) = 0$ then there exists representation of G into $C^\infty(\mathbb{R}^d)[[\hbar]]$ of the form (4.3). Moreover, if $H^1(G, C^\infty(\mathbb{R}^d)) = 0$, all these representations are equivalent.*

Proof. As a graded vector space $\mathbf{Pol}_d^\bullet(n)$ can be identified with the following direct sum with n -terms

$$C^\bullet(G, C^\infty(\mathbb{R}^d)) \oplus \dots \oplus C^\bullet(G, C^\infty(\mathbb{R}^d)),$$

since, for a cochain $P = \sum_{|\alpha| \leq n} f_\alpha(x) \xi^\alpha$, we have that $f_\alpha \in C^\bullet(G, C^\infty(\mathbb{R}^d))$ for all multi-indices α . Using Theorem 17, we only need to show that, in the case the action is trivial, d_1 respects this splitting. Let us compute the differential of $P \in \mathbf{Pol}_d^k(n)$:

$$\begin{aligned} (d_1 P)_{g_1, \dots, g_{k+1}} &= (dP)_{g_1, \dots, g_{k+1}} + 1_{g_1} \star P_{g_2, \dots, g_{k+1}} - (-1)^k P_{g_1, \dots, g_k} \star 1_{g_{k+1}}, \\ &= P_{g_2, \dots, g_{k+1}} + \sum_{i=1}^k (-1)^i P_{g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}} + (-1)^{k+1} P_{g_1, \dots, g_k}, \end{aligned}$$

since the product \star is now the standard product (associated with the standard quantization) because the action is trivial. Since only the f_α 's depend on the group variables, we obtain that

$$d_1 P = \sum_{|\alpha| \leq n} (\tilde{\delta} f_\alpha) \xi^\alpha,$$

where $\tilde{\delta}$ is the differential of the group cohomology of G in $C^\infty(\mathbb{R}^d)$ considered as a trivial bimodule. \square

Remark 19. This Theorem should be seen as a simplified version of Theorem 17 where existence and rigidity for general deformations of G -actions were discussed. Moreover, it should be compared with the analogue result of Pinzcon in [7].

5. CONCLUSIONS

This short note is a first of a series of papers whose main goal is the study of a family of deformations of G -actions obtained using a particular class of FIOs. The main tool used to define such a family of deformations is a DGA of formal amplitudes and its corresponding set of G -systems, see Definition 11. There are two main reasons to be interested in this family of deformations. The first can be found in the work of the first author, see [3], [4] and [5], where a new approach to quantization of Lagrangian submanifolds is proposed. The theory started in this note should represent the algebraic counterpart of the *micro-symplectic* approach to quantization developed in the cited papers.

Another motivation (not disjoint from the previous one) comes from the theory of the quantum momentum maps, see for example the paper [11]. As it is shown in [2] by

the authors of this letter, the G -systems can be used to get an explicit formula of the momentum map at the quantum level. In particular it is in this context that the main results of this paper about existence and uniqueness of formal deformations of a G -action find a different but quite natural interpretation (i.e. the existence of the corresponding quantum momentum map).

We conclude mentioning, without entering in the details which will be the main concern of future investigations, that to every (smooth) G -action φ it is also possible to associate a complex of *bounded* amplitudes $\mathcal{A}_\varphi^\bullet$. These amplitudes should be compared to the formal ones which were the main objects of this work. The construction of this new complex is formally the same as the one presented above, with the only difference that the *bounded cochains* will take now values in the the space $S_{2d}(1)$, see [6] for the definition of this class of symbols. This choice opens the interesting possibility of describing deformations of the *unitary* G -actions. In fact we observe that pseudo-differential operators with symbols in $S_{2d}(1)$ extend, thanks to the Calderon-Vaillancourt theorem, see for example [6] (from continuous operator on $\mathcal{S}(\mathbb{R}^d)$) to unitary operators to $L^2(\mathbb{R}^d)$.

Acknowledgments. We thank Ugo Bruzzo, Alberto Cattaneo, Giuseppe Dito, Domenico Fiorenza, Alberto Ibort, Gianni Landi, Marc Rieffel, Mauro Spreafico, Ali Tahzibi, Alan Weinstein and Sergio Zani as well as the hospitality of UC Berkeley and SISSA where part of this project was conducted. B.D. acknowledges support from FAPESP grant 2010/15069-8 and 2010/19365-0 and the University of São Paulo.

REFERENCES

- [1] C. A. Abad, M. Crainic, and B. Dherin, Tensor products of representations up to homotopy, *J. of Homotopy and Rel. Structures* **6**, 239–288(2011).
- [2] B. Dherin, I. Mencattini, Quantization of Momentum Maps and G -systems, pre-print, arXiv:1212.6489 (Unpublished yet).
- [3] A. Cattaneo, B. Dherin, A. Weinstein, Symplectic microgeometry I: micromorphisms, *J. Symplectic Geom.* **8**, 205–223 (2010).
- [4] A. Cattaneo, B. Dherin, A. Weinstein, Symplectic microgeometry II: generating functions. *Bull. Braz. Math. Soc. (N.S.)* **42**, (4), 507–536 (2011).
- [5] A. Cattaneo, B. Dherin, A. Weinstein, Symplectic microgeometry III: monoids, *J. Symplectic Geom.* **8**, 319–341 (2013).
- [6] A. Martinez, *An Introduction to Semiclassical and Microlocal Analysis*, Springer (2001).
- [7] G. Pinczon, Deformations of representations, *Lett. Math. Phys.* **1** (1977), 535–544.
- [8] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis III. Princeton University Press, 1993. xiv+695 pp.
- [9] F. Trèves, *Topological Vector Spaces, Distributions and Kernels* Dover Publications. (1995).
- [10] M. Zworski, *Semiclassical Analysis*, Graduate Studies in Mathematics **138**, AMS (2012).
- [11] P. Xu, Fedosov \ast -products and quantum momentum maps, *Commun. Math. Phys.* **197**, 167–197 (1998).

BENOIT DHERIN, DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840, USA

E-mail address: dherin@math.berkeley.edu

IGOR MENCATTINI, ICMC-USP UNIVERSIDADE DE SAO PAULO, AVENIDA TRABALHADOR SAO-CARLENSE 400 CENTRO, CEP: 13566-590, SAO CARLOS, SP, BRAZIL

E-mail address: `igorre@icmc.usp.br`